

CONTACT STRUCTURE ON MIXED LINKS

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ABSTRACT. A strongly non-degenerate mixed function has a Milnor open book structures on a sufficiently small sphere. We introduce the notion of a *holomorphic-like* mixed function and we will show that a link defined by such a mixed function has a canonical contact structure. Then we will show that this contact structure for a certain holomorphic-like mixed function is carried by the Milnor open book.

1. INTRODUCTION

Let $f(\mathbf{z})$ a holomorphic function with an isolated critical point at the origin. Then the Milnor fibration of f carries a canonical contact structure ([4, 2]). We consider a similar problem for mixed functions $f(\mathbf{z}, \bar{\mathbf{z}})$. We have shown that strongly non-degenerate mixed functions have Milnor fibrations on a small sphere [9]. However the situation is very different in the point that the tangent space of a mixed hypersurface is not a complex vector space. Therefore the restriction of the canonical contact structure need not give a contact structure on the mixed link. We introduce a class of mixed functions called *holomorphic-like* and we show that the restriction of the canonical contact structure gives a contact structure on the link (Theorem 13). A typical class of mixed functions we consider are given as the pull-back $g(\mathbf{w}, \bar{\mathbf{w}}) = \varphi_{a,b}^* f(\mathbf{w}, \bar{\mathbf{w}})$ of a convenient non-degenerate holomorphic function $f(\mathbf{z})$ by a homogeneous mixed covering $\varphi_{a,b} : \mathbb{C}^n \rightarrow \mathbb{C}^n$ which is defined by $\varphi_{a,b}(\mathbf{w}, \bar{\mathbf{w}}) = (w_1^a \bar{w}_1^b, \dots, w_n^a \bar{w}_n^b)$. Such a pull-back is a typical example of a holomorphic-like mixed function. Then we will show also that the Milnor open book is compatible with the canonical contact structures for these mixed functions (Theorem 22).

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2. PRELIMINARIES

2.1. Mixed functions and polar weightedness. Consider complex analytic function of $2n$ -variables $F(z_1, \dots, z_n, w_1, \dots, w_n)$ expanded in a convergent series $\sum_{\nu, \mu} c_{\nu, \mu} \mathbf{z}^\nu \mathbf{w}^\mu$ and consider the restriction $f(\mathbf{z}, \bar{\mathbf{z}})$ which is defined by the substitution $w_j = \bar{z}_j, j = 1, \dots, n$. We call this real analytic function *an analytic mixed function*. Namely $f(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{\nu, \mu} c_{\nu, \mu} \mathbf{z}^\nu \bar{\mathbf{z}}^\mu$

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where $\mathbf{z} = (z_1, \dots, z_n)$, $\bar{\mathbf{z}} = (\bar{z}_1, \dots, \bar{z}_n)$, $\mathbf{z}^\nu = z_1^{\nu_1} \cdots z_n^{\nu_n}$ for $\nu = (\nu_1, \dots, \nu_n)$ (respectively $\bar{\mathbf{z}}^\mu = \bar{z}_1^{\mu_1} \cdots \bar{z}_n^{\mu_n}$ for $\mu = (\mu_1, \dots, \mu_n)$). Here \bar{z}_i is the complex conjugate of z_i . Assume that f is a polynomial. Writing $z_j = x_j + i y_j$, it is easy to see that f is a polynomial of $2n$ -variables $x_1, y_1, \dots, x_n, y_n$. In this case, we call f a *mixed polynomial* of z_1, \dots, z_n .

A mixed polynomial $f(\mathbf{z}, \bar{\mathbf{z}})$ is called *polar weighted homogeneous* if there exist positive integers q_1, \dots, q_n and p_1, \dots, p_n and non-zero integers m_r, m_p such that

$$\begin{aligned} \gcd(q_1, \dots, q_n) = 1, \quad \gcd(p_1, \dots, p_n) = 1, \\ \sum_{j=1}^n q_j(\nu_j + \mu_j) = m_r, \quad \sum_{j=1}^n p_j(\nu_j - \mu_j) = m_p, \quad \text{if } c_{\nu, \mu} \neq 0 \end{aligned}$$

The weight vectors $Q = (q_1, \dots, q_n)$ and $P = (p_1, \dots, p_n)$ are called *the radial weight and the polar weight* respectively. Using radial weight and the polar weight, we define the radial $\mathbb{R}_{>0}$ -action and the polar S^1 -action as follows.

$$\begin{aligned} r \circ \mathbf{z} &= (r^{q_1} z_1, \dots, r^{q_n} z_n), \quad r \in \mathbb{R}_{>0} \\ e^{i\eta} \circ \mathbf{z} &= (e^{ip_1\eta} z_1, \dots, e^{ip_n\eta} z_n), \quad e^{i\eta} \in S^1 \end{aligned}$$

Then f satisfies the functional equalities

$$\begin{aligned} (1) \quad & f(r \circ (\mathbf{z}, \bar{\mathbf{z}})) = r^{m_r} f(\mathbf{z}, \bar{\mathbf{z}}), \quad r \in \mathbb{R}_{>0} \\ (2) \quad & f(e^{i\eta} \circ (\mathbf{z}, \bar{\mathbf{z}})) = e^{im_p\eta} f(\mathbf{z}, \bar{\mathbf{z}}), \quad e^{i\eta} \in S^1. \end{aligned}$$

These equalities give the following Euler equalities.

$$\begin{aligned} (3) \quad & \text{(Radial Euler equality)} : m_r f(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{i=1}^n q_i \left(\frac{\partial f}{\partial z_i} z_i + \frac{\partial f}{\partial \bar{z}_i} \bar{z}_i \right) \\ (4) \quad & \text{(Polar Euler equality)} : m_p f(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{i=1}^n p_i \left(\frac{\partial f}{\partial z_i} z_i - \frac{\partial f}{\partial \bar{z}_i} \bar{z}_i \right). \end{aligned}$$

We consider a special type of polar weighted homogeneous polynomial. A polar weighted homogeneous polynomial $f(\mathbf{z}, \bar{\mathbf{z}})$ is called *strongly polar weighted homogeneous* if the radial weight and the polar weights are the same, i.e., $p_j = q_j$ for $j = 1, \dots, n$. In this case, the radial and polar Euler equalities gives:

$$(5) \quad \begin{cases} \sum_{j=1}^n p_j z_j \frac{\partial f}{\partial z_j}(\mathbf{z}, \bar{\mathbf{z}}) = \frac{m_r + m_p}{2} f(\mathbf{z}, \bar{\mathbf{z}}), \\ \sum_{j=1}^n p_j \bar{z}_j \frac{\partial f}{\partial \bar{z}_j}(\mathbf{z}, \bar{\mathbf{z}}) = \frac{m_r - m_p}{2} f(\mathbf{z}, \bar{\mathbf{z}}) \end{cases}$$

The above equalities say that $f(\mathbf{z}, \bar{\mathbf{z}})$ is a weighted homogeneous polynomial for \mathbf{z} and $\bar{\mathbf{z}}$ independently. Furthermore $f(\mathbf{z}, \bar{\mathbf{z}})$ is called *strongly polar positive weighted homogeneous* if $\text{pdeg} f = m_p > 0$.

2.2. Euclidean metric and hermitian product. Recall that \mathbb{C}^n is canonically identified with \mathbb{R}^{2n} by $\mathbf{z} = (z_1, \dots, z_n) \mapsto \mathbf{z}_{\mathbb{R}} := (x_1, y_1, \dots, x_n, y_n) \in \mathbb{R}^{2n}$. The inner product in \mathbb{R}^{2n} is simply the real part of the hermitian product in \mathbb{C}^n . We denote the hermitian inner product as (\mathbf{z}, \mathbf{w}) for $\mathbf{z}, \mathbf{w} \in \mathbb{C}^n$

and the inner product as the vector in \mathbb{R}^{2n} as $(\mathbf{z}_{\mathbb{R}}, \mathbf{w}_{\mathbb{R}})_{\mathbb{R}}$. Namely putting $\mathbf{w} = (w_1, \dots, w_n)$ with $w_j = u_j + iv_j$,

$$(\mathbf{z}, \mathbf{w}) = \sum_{j=1}^n z_j \bar{w}_j, \quad (\mathbf{z}_{\mathbb{R}}, \mathbf{w}_{\mathbb{R}})_{\mathbb{R}} = \sum_{j=1}^n (x_j u_j + y_j v_j).$$

Thus $\Re(\mathbf{z}, \mathbf{w}) = (\mathbf{z}_{\mathbb{R}}, \mathbf{w}_{\mathbb{R}})_{\mathbb{R}}$. By the triviality of the tangent bundle $T_p \mathbb{R}^{2n}$, we identify $T_p \mathbb{R}^{2n}$ with $\mathbb{R}^{2n} = \mathbb{C}^n$. Thus $\sum_{j=1}^n (x_j (\frac{\partial}{\partial x_j})_p + y_j (\frac{\partial}{\partial y_j})_p)$ is identified with $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{C}^n$ and $z_j = x_j + i y_j$. Recall the complexified tangent vectors are defined by

$$\frac{\partial}{\partial z_j} = \frac{1}{2} \left(\frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right), \quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2} \left(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right)$$

Thus a complex vector $\mathbf{z} \in \mathbb{C}^{2n} = \mathbb{R}^{2n}$ is identified with the tangent vector

$$\sum_{j=1}^n (x_j \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial y_j}) = \sum_{j=1}^n (z_j \frac{\partial}{\partial z_j} + \bar{z}_j \frac{\partial}{\partial \bar{z}_j}).$$

$J : T\mathbb{C}^n \rightarrow T\mathbb{C}^n$ is the almost complex structure defined by

$$\begin{aligned} J\left(\frac{\partial}{\partial x_j}\right) &= \frac{\partial}{\partial y_j}, & J\left(\frac{\partial}{\partial y_j}\right) &= -\frac{\partial}{\partial x_j} \\ J\left(\frac{\partial}{\partial z_j}\right) &= i \frac{\partial}{\partial \bar{z}_j}, & J\left(\frac{\partial}{\partial \bar{z}_j}\right) &= -i \frac{\partial}{\partial z_j}. \end{aligned}$$

For a real valued mixed function $h(\mathbf{z}, \bar{\mathbf{z}})$, we define *the real gradient* $\text{grad}_{\mathbb{R}} h \in \mathbb{R}^{2n}$ (or *Riemannian gradient* in [2]) as

$$\text{grad}_{\mathbb{R}} h(\mathbf{z}, \bar{\mathbf{z}}) = \left(\frac{\partial h}{\partial x_1}(\mathbf{z}, \bar{\mathbf{z}}), \frac{\partial h}{\partial y_1}(\mathbf{z}, \bar{\mathbf{z}}), \dots, \frac{\partial h}{\partial x_n}(\mathbf{z}, \bar{\mathbf{z}}), \frac{\partial h}{\partial y_n}(\mathbf{z}, \bar{\mathbf{z}}) \right) \in \mathbb{R}^{2n}.$$

We define also *the complex gradient of h* or *hermitian gradient* in [2] as follows.

$$\nabla h(\mathbf{z}, \bar{\mathbf{z}}) = 2 \left(\frac{\partial h}{\partial z_1}(\mathbf{z}, \bar{\mathbf{z}}), \dots, \frac{\partial h}{\partial z_n}(\mathbf{z}, \bar{\mathbf{z}}) \right) \in \mathbb{C}^n.$$

Proposition 1. *Assume that $h(\mathbf{z}, \bar{\mathbf{z}})$ is a real valued mixed function and let $\mathbf{z}(t) = (z_1(t), \dots, z_n(t))$, $z_j(t) = x_j(t) + i y_j(t)$, $-1 \leq t \leq 1$ be a smooth curve in \mathbb{C}^n and let $\mathbf{z}(0) = \mathbf{u}$, $\frac{d\mathbf{z}}{dt}(0) = \mathbf{v}$. Then we have*

$$\frac{dh(\mathbf{z}(t), \bar{\mathbf{z}}(t))}{dt}(0) = (\mathbf{v}_{\mathbb{R}}, \text{grad}_{\mathbb{R}} h(\mathbf{u}, \bar{\mathbf{u}}))_{\mathbb{R}} = \Re(\mathbf{v}, \nabla h(\mathbf{u}, \bar{\mathbf{u}})).$$

Proof. The second equality follows from the simple calculation:

$$\begin{aligned} \frac{dh(\mathbf{z}(t), \bar{\mathbf{z}}(t))}{dt}(0) &= \sum_{i=1}^n v_i \frac{\partial h}{\partial z_i}(\mathbf{z}_0, \bar{\mathbf{z}}_0) + \sum_{i=1}^n \bar{v}_i \frac{\partial h}{\partial \bar{z}_i}(\mathbf{z}_0, \bar{\mathbf{z}}_0) \\ &= \sum_{i=1}^n v_i \frac{\partial h}{\partial z_i}(\mathbf{z}_0, \bar{\mathbf{z}}_0) + \sum_{i=1}^n \bar{v}_i \frac{\partial \bar{h}}{\partial \bar{z}_i}(\mathbf{z}_0, \bar{\mathbf{z}}_0) \\ &= \Re(\mathbf{v}, \nabla h(\mathbf{z}_0, \bar{\mathbf{z}}_0)). \end{aligned}$$

□

Thus the tangent space of the real hypersurface $H := h^{-1}(0)$ at a smooth point \mathbf{z}_0 is given by

$$\begin{aligned} T_{\mathbf{z}_0}H &= \{\mathbf{u} \in \mathbb{R}^{2n} \mid (\mathbf{u}, \text{grad}_{\mathbb{R}} h(\mathbf{z}_0, \bar{\mathbf{z}}_0))_{\mathbb{R}} = 0\} \\ &= \{\mathbf{w} \in \mathbb{C}^n \mid \Re(\mathbf{w}, \nabla h(\mathbf{z}_0, \bar{\mathbf{z}}_0)) = 0\}. \end{aligned}$$

For our later purpose, it is more convenient to use the hermitian gradient.

2.2.1. Holomorphic function case. Assume that $f(\mathbf{z})$ is a holomorphic function defined on a neighborhood of the origin. Then the hermitian gradient ∇f is defined by (see [6, 2])

$$\nabla f(\mathbf{z}) = \left(\frac{\overline{\partial f}}{\partial z_1}, \dots, \frac{\overline{\partial f}}{\partial z_n} \right).$$

Consider the complex hypersurface $V = f^{-1}(0) \subset \mathbb{C}^n$.

Proposition 2. *Assume that $p \in V$ is a regular point. Then the tangent space $T_p V$ is the complex subspace given by*

$$T_p V = \{\mathbf{v} \in \mathbb{C}^n \mid (\mathbf{v}, \nabla f(\mathbf{z})) = 0\}.$$

Remark 3. *Let $f(\mathbf{z}, \bar{\mathbf{z}})$ be a complex valued mixed function and write $f(\mathbf{z}, \bar{\mathbf{z}}) = g(\mathbf{z}, \bar{\mathbf{z}}) + i h(\mathbf{z}, \bar{\mathbf{z}})$. Consider a mixed hypersurface $V = f^{-1}(0)$ and assume that $p \in V$ is a mixed regular point. Then the tangent space $T_p V$ has no complex structure in general and there does not exist a single gradient vector to describe $T_p V$. It is described by two hermitian gradient vectors as*

$$T_p V = \{\mathbf{v} \in \mathbb{C}^n \mid \Re(\mathbf{v}, \nabla g(p, \bar{p})) = \Re(\mathbf{v}, \nabla h(p, \bar{p})) = 0\}.$$

2.2.2. Weighted spheres. For a given positive integer vector $\mathbf{a} = (a_1, \dots, a_n)$ and a positive number $r > 0$, we consider

$$\rho_{\mathbf{a}}(\mathbf{z}) = \sum_{j=1}^n a_j |z_j|^2 = \sum_{j=1}^n a_j (x_j^2 + y_j^2)$$

and we define the weighted sphere $\mathbb{S}_r(\mathbf{a})$ by

$$\mathbb{S}_r(\mathbf{a}) := \{\mathbf{z} \in \mathbb{C}^n \mid \rho_{\mathbf{a}}(\mathbf{z}) = r^2\}.$$

The standard sphere is defined by the weight vector $\mathbf{a} = (1, \dots, 1)$ and in this case, we simply write \mathbb{S}_r . Put $\tilde{\mathbf{z}}(\mathbf{a}) = (a_1 z_1, \dots, a_n z_n)$. Then $\nabla \rho_{\mathbf{a}}(\mathbf{z}) = 2\tilde{\mathbf{z}}(\mathbf{a})$. Therefore the tangent space at $\mathbf{z}_0 \in \mathbb{S}_r(\mathbf{a})$ is given by

$$T_{\mathbf{z}_0} \mathbb{S}_r(\mathbf{a}) = \{\mathbf{w} \mid \Re(\mathbf{w}, \tilde{\mathbf{z}}_0(\mathbf{a})) = 0\}, \quad T_{\mathbf{z}_0} \mathbb{S}_r = \{\mathbf{w} \mid \Re(\mathbf{w}, \mathbf{z}_0) = 0\}.$$

2.2.3. Transversality of a polar weighted homogeneous hypersurface. Let f be a polar weighted homogeneous polynomial of radial weight type $(q_1, \dots, q_n; m_r)$ and of polar weight type $(p_1, \dots, p_n; m_p)$. Let $V = f^{-1}(0)$ and write $f(\mathbf{z}, \bar{\mathbf{z}}) = h(\mathbf{z}, \bar{\mathbf{z}}) + ig(\mathbf{z}, \bar{\mathbf{z}})$ with real valued mixed functions h, g .

Proposition 4. (*Transversality*) Assume that V has an isolated mixed singularity at the origin. Then the sphere $\mathbb{S}_r(\mathbf{a}) = \{\mathbf{z} \in \mathbb{C}^n; \rho_{\mathbf{a}}(\mathbf{z}) = r^2\}$ intersects transversely with V for any $r > 0$.

Proof. The proof is the exact same as that of Proposition 4, [8]. Assume that $\mathbf{z}_0 \in S_{\mathbf{a}}(r) \cap V$ is a point where the sphere is not transverse. Note that the tangent space is the real orthogonal space to two hermitian gradient vectors $\nabla h(\mathbf{z}_0, \bar{\mathbf{z}}_0)$ and $\nabla g(\mathbf{z}_0, \bar{\mathbf{z}}_0)$. Let $\rho_{\mathbf{a}}(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{j=1}^n a_j |z_j|^2$. The non-transversality implies for example, there is a linear relation

$$(6) \quad \nabla \rho_{\mathbf{a}}(\mathbf{z}_0, \bar{\mathbf{z}}_0) = \alpha \nabla h(\mathbf{z}_0, \bar{\mathbf{z}}_0) + \beta \nabla g(\mathbf{z}_0, \bar{\mathbf{z}}_0)$$

with some $\alpha, \beta \in \mathbb{R}$. We consider the radial orbit curve $\mathbf{z}(t) = t \circ \mathbf{z}_0 = (t^{q_1} z_{01}, \dots, t^{q_n} z_{0n})$. The tangent vector $\frac{d\mathbf{z}}{dt}(1) = \tilde{\mathbf{z}}_0(\mathbf{q}) = (q_1 z_{01}, \dots, q_n z_{0n})$ with $\mathbf{q} = (q_1, \dots, q_n)$. Then we have an inequality:

$$(7) \quad \left. \frac{d\rho_{\mathbf{a}}(\mathbf{z}(t))}{dt} \right|_{t=1} = \Re(\tilde{\mathbf{z}}_0(\mathbf{q}), \nabla \rho_{\mathbf{a}}(\mathbf{z}_0, \bar{\mathbf{z}}_0)) = \Re(\tilde{\mathbf{z}}_0(\mathbf{q}), 2\tilde{\mathbf{z}}_0(\mathbf{a})) > 0$$

On the other hand, the mixed real polynomials $h(\mathbf{z}, \bar{\mathbf{z}}), g(\mathbf{z}, \bar{\mathbf{z}})$ are radially weighted homogeneous under the same weight $\mathbf{q} = (q_1, \dots, q_n)$. This implies $h(\mathbf{z}(t)) \equiv g(\mathbf{z}(t)) \equiv 0$ and we have two equalities:

$$\begin{aligned} \left. \frac{dh(\mathbf{z}(t))}{dt} \right|_{t=1} &= \Re(\tilde{\mathbf{z}}_0(\mathbf{q}), \nabla h(\mathbf{z}_0, \bar{\mathbf{z}}_0)) = 0, \\ \left. \frac{dg(\mathbf{z}(t))}{dt} \right|_{t=1} &= \Re(\tilde{\mathbf{z}}_0(\mathbf{q}), \nabla g(\mathbf{z}_0, \bar{\mathbf{z}}_0)) = 0. \end{aligned}$$

Now we have a contradiction to (7) by (6):

$$\begin{aligned} 0 &< \Re(\tilde{\mathbf{z}}_0(\mathbf{q}), \nabla \rho_{\mathbf{a}}(\mathbf{z}_0, \bar{\mathbf{z}}_0)) = \\ &\quad \alpha \Re(\tilde{\mathbf{z}}_0(\mathbf{q}), \nabla h(\mathbf{z}_0, \bar{\mathbf{z}}_0)) + \beta \Re(\tilde{\mathbf{z}}_0(\mathbf{q}), \nabla g(\mathbf{z}_0, \bar{\mathbf{z}}_0)) = 0. \end{aligned}$$

□

2.3. Mixed functions of strongly polar weighted homogeneous face type. Consider a mixed function $f(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{\nu, \mu} c_{\nu\mu} \mathbf{z}^\nu \bar{\mathbf{z}}^\mu$. Recall that for a weight vector $P = (p_1, \dots, p_n)$, the face function f_P is defined by the linear sum of the monomials with the radial degree is the minimum ([9]). Thus $f_P(\mathbf{z}, \bar{\mathbf{z}})$ is a radially weighted homogeneous polynomial with the weight P .

2.3.1. Definition. f is called a *mixed function of polar positive weighted homogeneous face type* if for any weight vector P with $\dim \Delta(P) = n - 1$, the face function $f_P(\mathbf{z}, \bar{\mathbf{z}})$ is a polar weighted homogeneous polynomial with some weight vector P' (P' need not be P) and $\text{pdeg}_{P'} f_P > 0$.

f is called a *mixed function of strongly polar positive weighted homogeneous face type* if the face function $f_P(\mathbf{z}, \bar{\mathbf{z}})$ is a strongly polar positive

weighted homogeneous polynomial with the same weight vector P , for any P with $\dim \Delta(P) = n - 1$.

Proposition 5. (1) Assume that $f(\mathbf{z}, \bar{\mathbf{z}})$ is a convenient mixed function of polar positive weighted homogeneous face type. Then for any weight vector P , f_P is also polar weighted homogeneous polynomial.

(2) Assume that $f(\mathbf{z}, \bar{\mathbf{z}})$ is a convenient mixed function of strongly polar positive weighted homogeneous face type. Then for any weight vector P , f_P is also a strongly polar positive weighted homogeneous polynomial.

Proof. The assertion (1) is obvious, as any face Δ of $\Gamma(f)$ is a subface of a face of dimension $n - 1$. We consider the assertion (2). The assertion is proved by the descending induction on $\dim \Delta(P)$. The assertion for the case $\dim \Delta(P) = n - 1$ is the definition itself. Suppose that $\dim \Delta(P) = k$ and the assertion is true for faces with $\dim \Delta \geq k + 1$. In the dual Newton diagram, P is contained in the interior of a cell Ξ whose vertices Q satisfies $\dim \Delta(Q) \geq k + 1$. This implies P is a linear combination $\sum_{j=1}^s a_j Q_j$ with $a_j \geq 0$ and $\dim \Delta(Q_j) \geq k + 1$ where Q_1, \dots, Q_s are vertices of Ξ . This implies also that $\Delta(P) = \cap_j^s \Delta(Q_j)$. Write $f_P(\mathbf{z}, \bar{\mathbf{z}}) = \sum_k c_k \mathbf{z}^{\nu_k} \bar{\mathbf{z}}^{\mu_k}$. As f_{Q_j} is a strongly polar weighted homogeneous polynomial with weight Q_j ,

$$\text{pdeg}_{Q_j} \mathbf{z}^{\nu_k} \bar{\mathbf{z}}^{\mu_k} = m_j, \quad j = 1, \dots, s$$

where m_j is independent of k . This implies f_P is polar weighted homogeneous polynomial of weight P with polar degree $\sum_{j=1}^s a_j m_j > 0$. \square

As an obvious but important example, we have

Proposition 6. A holomorphic function $f(\mathbf{z}, \bar{\mathbf{z}})$ is a mixed function of strongly polar positive weighted homogeneous face type.

A mixed function of strongly polar weighted homogeneous face type behaves like a non-degenerate holomorphic function. In [7], we have proved a Varchenko type formula for the zeta function.

2.4. Mixed cyclic covering. Consider two non-negative integer vectors $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$. We say \mathbf{a} is *strictly bigger* than \mathbf{b} if $a_j > b_j \geq 0$ for any $j = 1, \dots, n$. If this is the case, we denote it as $\mathbf{a} \gg \mathbf{b}$. For given \mathbf{a}, \mathbf{b} with $\mathbf{a} \gg \mathbf{b}$, we consider real analytic mapping $\varphi_{\mathbf{a}, \mathbf{b}}$:

$$\varphi_{\mathbf{a}, \mathbf{b}} : \mathbb{C}^n \rightarrow \mathbb{C}^n, \quad \varphi_{\mathbf{a}, \mathbf{b}}(\mathbf{w}) = (w_1^{a_1} \bar{w}_1^{b_1}, \dots, w_n^{a_n} \bar{w}_n^{b_n}).$$

We call $\varphi_{\mathbf{a}, \mathbf{b}}$ a *mixed cyclic covering mapping* associated with integer vectors $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$. In fact, over \mathbb{C}^{*n} , $\varphi_{\mathbf{a}, \mathbf{b}} : \mathbb{C}^{*n} \rightarrow \mathbb{C}^{*n}$ is a $\prod_{j=1}^n (a_j - b_j)$ -fold polycyclic covering.

We say that $\varphi_{\mathbf{a}, \mathbf{b}}$ is *homogeneous* if $\mathbf{a} = (a, \dots, a)$ and $\mathbf{b} = (b, \dots, b)$ where a, b are integers such that $a > b \geq 0$. In this case, we denote $\varphi_{a, b}$

instead of $\varphi_{\mathbf{a}, \mathbf{b}}$ and we call $\varphi_{a,b}$ a homogeneous mixed covering. For a given mixed function $f(\mathbf{z}, \bar{\mathbf{z}})$, the pull-back $g = \varphi_{\mathbf{a}, \mathbf{b}}^*(f)$ is defined by

$$g(\mathbf{w}, \bar{\mathbf{w}}) = f \circ \varphi_{\mathbf{a}, \mathbf{b}}(\mathbf{w}, \bar{\mathbf{w}}) = f(w_1^{a_1} \bar{w}_1^{b_1}, \dots, w_n^{a_n} \bar{w}_n^{b_n}).$$

Proposition 7. *Let $f(\mathbf{z}, \bar{\mathbf{z}})$ be a non-degenerate convenient mixed function of polar weighted homogeneous face type. Let $\varphi = \varphi_{\mathbf{a}, \mathbf{b}}$ be the mixed cyclic covering associated with $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$ as above. Consider the pull-back $g(\mathbf{w}, \bar{\mathbf{w}}) = f(\varphi(\mathbf{w}, \bar{\mathbf{w}}))$. Then $g(\mathbf{w}, \bar{\mathbf{w}})$ is a convenient non-degenerate mixed function of polar weighted homogeneous face type.*

If f is of strongly polar positive weighted homogeneous face type and $\varphi = \varphi_{a,b}$ is a homogeneous mixed covering mapping, g is also of strongly polar positive weighted homogeneous face type.

Proof. Let P be a weight vector and consider $f_P(\mathbf{z}, \bar{\mathbf{z}})$. It is a radially weighted homogeneous polynomial under the weight P . Let $R = (r_1, \dots, r_n)$ be the polar weight of f_P . Let d_r and d_p be the radial and polar degree of f_P . We consider the normalized weight $Q = (q_1, \dots, q_n) \in \mathbb{Q}^n$ and $S = (s_1, \dots, s_n) \in \mathbb{Q}^n$ where $q_j = p_j/d_r$ and $s_j = r_j/d_p$. We consider also the normalized weights $\hat{Q} = (\hat{q}_1, \dots, \hat{q}_n)$ and $\hat{S} = (\hat{s}_1, \dots, \hat{s}_n)$ where

$$\hat{q}_j = q_j/(a_j + b_j), \quad \hat{s}_j = s_j/(a_j - b_j), \quad j = 1, \dots, n.$$

Consider a monomial $M = z_1^{m_1} \bar{z}_1^{\ell_1} \dots z_n^{m_n} \bar{z}_n^{\ell_n}$ in f_P , i.e. $\deg_Q M = 1$, $\text{pdeg}_S M = 1$. Consider the pull-back of M ,

$$M' = \varphi^* M = \prod_{j=1}^n (w_j^{a_j} \bar{w}_j^{b_j})^{m_j} (\bar{w}_j^{a_j} w_j^{b_j})^{\ell_j}$$

Then by an easy calculation, we have

$$\deg_{\hat{Q}} M' = \sum_{j=1}^n q_j(m_j + \ell_j) = \deg_Q M = 1$$

$$\text{pdeg}_{\hat{S}} M' = \sum_{j=1}^n s_j(m_j - \ell_j) = \text{pdeg}_S M = 1$$

This implies that $\varphi^* f_P = g_{\hat{Q}}$ is a radially weighted homogeneous polynomial by the normalized weight vector \hat{Q} and $\varphi^* f_P$ is a polar weighted homogeneous polynomial by the normal weight vector \hat{S} . Non-degeneracy is the result of the commutative diagram:

$$\begin{array}{ccc} \mathbb{C}^{*n} & \xrightarrow{\varphi} & \mathbb{C}^{*n} \\ \downarrow g_{\hat{Q}} & & \downarrow f_Q \\ \mathbb{C} & = & \mathbb{C} \end{array}$$

We observe that if $\varphi = \varphi_{a,b}$ and $p_j = r_j$,

$$\hat{s}_j(a - b)d_p = r_j = p_j = \hat{q}_j(a + b)d_r.$$

which implies that $\varphi^* f_P$ is strongly polar weighted homogeneous. \square

As holomorphic functions are obviously mixed functions of strongly polar weighted homogeneous face type, we have:

Corollary 8. *Assume that $f(\mathbf{z})$ is a convenient non-degenerate holomorphic function and $g(\mathbf{w}, \bar{\mathbf{w}}) = \varphi^* f(\mathbf{w}, \bar{\mathbf{w}})$ with $\varphi = \varphi_{\mathbf{a}, \mathbf{b}}$. Then $g(\mathbf{w}, \bar{\mathbf{w}})$ is a convenient non-degenerate mixed function of polar weighted homogeneous face type. If further $\varphi = \varphi_{a, b}$, homogeneous with $a > b \geq 0$, g is of strongly polar positive weighted homogeneous face type.*

3. CONTACT STRUCTURE

3.1. Contact structure and a contact submanifold of a sphere. Let M be a smooth oriented manifold of dimension $2n - 1$. A *contact structure* on M is a hyperplane distribution ξ in the tangent bundle TM ($M \ni p \mapsto \xi(p) \subset T_p M$) which is induced by a global 1-form α by $\xi(p) = \text{Ker } \alpha$ such that $\alpha \wedge (d\alpha)^{n-1}$ is nowhere vanishing $(2n - 1)$ form. We say α is *positive* if $\alpha \wedge (d\alpha)^{n-1}$ is a positive form.

We consider the radius function $\rho(\mathbf{z}, \bar{\mathbf{z}}) = z_1 \bar{z}_1 + \cdots + z_n \bar{z}_n$. The level manifold $\rho^{-1}(r^2)$ is nothing but the sphere \mathbb{S}_r . On \mathbb{S}_r , we consider the canonical contact structure ξ defined by the contact form $\alpha := -d^c \rho = -d\rho \circ J$ where J is the complex structure. More explicitly,

$$\alpha = \sum_{j=1}^n -i(\bar{z}_j dz_j - z_j d\bar{z}_j) = 2 \sum_{j=1}^n (x_j dy_j - y_j dx_j).$$

$\xi(\mathbf{z})$ is nothing but the complex hyperplane which is hermitian orthogonal to \mathbf{z} : $\xi(\mathbf{z}) = \{\mathbf{v} \mid (\mathbf{v}, \mathbf{z}) = 0\}$.

Let $\omega = d\alpha = -dd^c \varphi$. Then ω is explicitly written as

$$\omega(\mathbf{z}) = 2i \sum_{j=1}^n dz_j \wedge \bar{d}z_j = 4 \sum_{j=1}^n dx_j \wedge dy_j$$

and ω defines a symplectic structure on ξ . We have a canonical equality ([2]):

$$(8) \quad 4 \Re(\mathbf{u}, \mathbf{v}) = \omega(\mathbf{u}, J\mathbf{v}), \quad \mathbf{u}, \mathbf{v} \in T\mathbb{S}_r.$$

The *Reeb vector field* $R \in \Gamma(\mathbb{S}_r, T\mathbb{S}_r)$ is defined by the property:

$$\alpha(R) = 1, \quad \iota_R(\omega) = 0.$$

Here ι_R is the inner derivative by R . In our case,

$$\begin{aligned} R(\mathbf{z}) &= \frac{i\mathbf{z}}{2\rho(\mathbf{z})}, \quad \text{or as a tangent vector} \\ &= \frac{i}{2\rho(\mathbf{z})} \sum_{j=1}^n (z_j \frac{\partial}{\partial z_j} - \bar{z}_j \frac{\partial}{\partial \bar{z}_j}) = \frac{1}{2r^2} \sum_{j=1}^n (x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j}). \end{aligned}$$

We consider a real codimension two submanifold $K \subset \mathbb{S}_r$. We say K is a (positive) contact submanifold of \mathbb{S}_r if the restriction $\alpha|_K$ defines a contact submanifold, i.e. $(2n-3)$ -form $\alpha \wedge (d\alpha)^{n-2}$ is nowhere vanishing form (respectively positive form) of K .

3.2. Remarks on the orientation. The orientation of \mathbb{S}_r is given as follows. A $(2n-1)$ -form Ω is positive if and only if $d\rho \wedge \Omega$ is a positive form of \mathbb{C}^n . Thus $\alpha \wedge (d\alpha)^{n-1}$ is positive. Let $f(\mathbf{z}, \bar{\mathbf{z}}) = g(\mathbf{z}, \bar{\mathbf{z}}) + i h(\mathbf{z}, \bar{\mathbf{z}})$ be a non-degenerate mixed function with an isolated mixed singularity at the origin. Let $K_r = f^{-1}(0) \cap \mathbb{S}_r$ with a sufficiently small r . The orientation of K_r is given by an $(2n-3)$ form Ω' such that $d\rho \wedge \Omega' \wedge dg \wedge dh$ is a positive form of \mathbb{C}^n .

3.3. Contact structure on mixed links. First we prepare a lemma. Put $f(\mathbf{z}, \bar{\mathbf{z}}) = g(\mathbf{z}, \bar{\mathbf{z}}) + i h(\mathbf{z}, \bar{\mathbf{z}})$, where g, h are real valued mixed functions. We use hereafter the following notation for simplicity.

$$f_{z_j} = \frac{\partial f}{\partial z_j}, \quad f_{\bar{z}_j} = \frac{\partial f}{\partial \bar{z}_j}.$$

Lemma 9. (1) $d\rho \wedge \alpha$ is given as follows.

$$d\rho \wedge \alpha = i \sum_{a,b=1}^n A_{a,\bar{b}} dz_a \wedge d\bar{z}_b, \quad A_{a,\bar{b}} = 2\bar{z}_a z_b.$$

(2) The two form $dg \wedge dh$ can be written as follows.

$$dg \wedge dh = i \sum_{a,b=1}^n B_{a,\bar{b}} dz_a \wedge d\bar{z}_b + R$$

where

$$B_{a,\bar{b}} = \frac{1}{2} (f_{z_a} \overline{f_{z_b}} - \overline{f_{\bar{z}_a}} f_{\bar{z}_b})$$

R is a linear combination of two forms $dz_a \wedge dz_b$ and $d\bar{z}_a \wedge d\bar{z}_b$.

Proof. The assertion (1) is a result of a simple calculation:

$$d\rho \wedge \alpha = \left(\sum_{j=1}^n (z_j d\bar{z}_j + \bar{z}_j dz_j) \right) \wedge \left(i \sum_{k=1}^n (z_k d\bar{z}_k - \bar{z}_k dz_k) \right).$$

For (2), we use the equality

$$g = \frac{1}{2}(f + \bar{f}), \quad h = \frac{-i}{2}(f - \bar{f}).$$

Thus we have

$$\begin{aligned} dg &= \frac{1}{2} \sum_{j=1}^n \{ (f_{z_j} + \bar{f}_{z_j}) dz_j + (f_{\bar{z}_j} + \bar{f}_{\bar{z}_j}) d\bar{z}_j \} \\ dh &= \frac{-i}{2} \sum_{j=1}^n \{ (f_{z_j} - \bar{f}_{z_j}) dz_j + (f_{\bar{z}_j} - \bar{f}_{\bar{z}_j}) d\bar{z}_j \} \end{aligned}$$

As $\bar{f}_{\bar{z}_j} = \overline{f_{z_j}}$ and $\bar{f}_{z_j} = \overline{f_{\bar{z}_j}}$, the assertion follows by a simple calculation. \square

Corollary 10. *The four form $d\rho \wedge \alpha \wedge dg \wedge dh$ is given as follows.*

$$d\rho \wedge \alpha \wedge dg \wedge dh = - \sum_{a,b=1}^n C_{a,b} dz_a \wedge d\bar{z}_a \wedge dz_b \wedge d\bar{z}_b + S$$

$$C_{a,b} = |\bar{z}_a f_{z_b} - \bar{z}_b f_{z_a}|^2 - |z_a f_{\bar{z}_b} - z_b f_{\bar{z}_a}|^2$$

where S is a linear combination of other type of four forms.

Proof. Write

$$d\rho \wedge \alpha \wedge dg \wedge dh = - \sum_{a,b=1}^n C_{a,b} dz_a \wedge d\bar{z}_a \wedge dz_b \wedge d\bar{z}_b + S.$$

Then by Lemma 9, we have

$$\begin{aligned} C_{a,b} &= A_{a,\bar{a}} B_{b,\bar{b}} + A_{b,\bar{b}} B_{a,\bar{a}} - A_{a,\bar{b}} B_{b,\bar{a}} - A_{b,\bar{a}} B_{a,\bar{b}} \\ &= |z_a|^2 (|f_{z_b}|^2 - |f_{\bar{b}}|^2) + |z_b|^2 (|f_{z_a}|^2 - |f_{\bar{a}}|^2) \\ &\quad - 2\bar{z}_a z_b (f_{z_b} \overline{f_{z_a}} - \overline{f_{\bar{z}_b}} f_{\bar{a}}) - 2\bar{z}_b z_a (f_{z_a} \overline{f_{z_b}} - \overline{f_{\bar{z}_a}} f_{\bar{b}}) \\ &= (z_a \overline{f_b} - z_b \overline{f_a})(\bar{z}_a f_b - \bar{z}_b f_a) - (z_b \overline{f_a} - z_a \overline{f_b})(\bar{z}_b f_a - \bar{z}_a f_b) \\ &= |\bar{z}_a f_{z_b} - \bar{z}_b f_{z_a}|^2 - |z_a f_{\bar{z}_b} - z_b f_{\bar{z}_a}|^2. \end{aligned}$$

\square

Define $C(\mathbf{z}, \bar{\mathbf{z}}) := \sum_{1 \leq a < b \leq n} C_{a,b}(\mathbf{z}, \bar{\mathbf{z}})$. By Corollary 10 and an easy computation gives the following.

Corollary 11. *We have*

$$\begin{aligned} &d\rho \wedge \alpha \wedge d\alpha^{n-2} \wedge dg \wedge dh(\mathbf{z}, \bar{\mathbf{z}}) \\ &= i^n 2^{n-2} (n-2)! C(\mathbf{z}, \bar{\mathbf{z}}) dz_1 \wedge \bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n \\ &= 4^{n-1} (n-2)! C(\mathbf{z}, \bar{\mathbf{z}}) dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n. \end{aligned}$$

3.4. Holomorphic-like mixed function. Let U be an open neighborhood of the origin. A mixed function $f(\mathbf{z}, \bar{\mathbf{z}})$ which is defined on U with an isolated mixed singularity at the origin is called *holomorphic-like* (respectively *anti-holomorphic-like*) if for any $\mathbf{z} \in f^{-1}(0) \cap U$,

$$(9) \quad C(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{1 \leq a < b \leq n} C_{a,b} \geq 0$$

$$(10) \quad C_{a,b} = |z_a \overline{f_{z_b}} - z_b \overline{f_{z_a}}|^2 - |z_a f_{\bar{z}_b} - z_b f_{\bar{z}_a}|^2$$

(or respectively $C(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{1 \leq a < b \leq n} C_{a,b} \leq 0$).

We say that $f(\mathbf{z}, \bar{\mathbf{z}})$ is *strictly holomorphic-like* (resp. *strictly anti-holomorphic-like*) in U if $C(\mathbf{z}, \bar{\mathbf{z}}) > 0$ (resp. $C(\mathbf{z}, \bar{\mathbf{z}}) < 0$) on any smooth point $\mathbf{z} \in U \cap f^{-1}(0) \setminus \{\mathbf{0}\}$.

Remark. If f is a holomorphic function, $f_{\bar{z}_j} = 0$ and $C_{a,b} \geq 0$ for any $1 \leq a < b \leq n$. Thus $f(\mathbf{z})$ is obviously (strictly) holomorphic-like. If $f(\bar{\mathbf{z}})$ is an anti-holomorphic function, $f_{z_j} = 0$ and $f(\mathbf{z})$ is anti-holomorphic-like.

Lemma 12. *Assume that $f(\mathbf{z})$ is a holomorphic function and let $g(\mathbf{w}, \bar{\mathbf{w}}) = \varphi^* f(\mathbf{w}, \bar{\mathbf{w}}) = f(\varphi(\mathbf{w}, \bar{\mathbf{w}}))$ where φ is a mixed homogeneous cyclic covering associated with integers $a > b \geq 0$ (respectively $0 \leq a < b$). Then g is a holomorphic-like (resp. anti-holomorphic-like) mixed function in a neighborhood of the origin.*

Proof. By an easy calculation, we get

$$\begin{aligned} g_{w_j} &= f_{z_j}(\varphi(\mathbf{w})) a w_j^{a-1} \bar{w}_j^b, \\ g_{\bar{w}_j} &= f_{z_j}(\varphi(\mathbf{w})) b w_j^a \bar{w}_j^{b-1} \end{aligned}$$

and for the case $b \geq 1$ we get

$$\begin{aligned} C_{j,k} &= |a \bar{w}_j w_k^{a-1} \bar{w}_k^b \varphi^* f_{z_k} - a \bar{w}_k w_j^{a-1} \bar{w}_j^b \varphi^* f_{z_j}|^2 \\ &\quad - |b w_j w_k^a \bar{w}_k^{b-1} \varphi^* f_{z_k} - b w_k w_j^a \bar{w}_j^{b-1} \varphi^* f_{z_j}|^2 \\ &= (a^2 - b^2) |w_j w_k|^2 \left(w_k^{a-1} \bar{w}_k^{b-1} \varphi^* f_{z_k} - w_j^{a-1} \bar{w}_j^{b-1} \varphi^* f_{z_j} \right)^2 \end{aligned}$$

If $b = 0$, g is a holomorphic function and

$$C_{j,k} = a^2 |\bar{w}_j w_k^{a-1} \varphi^* f_{z_k} - \bar{w}_k w_j^{a-1} \varphi^* f_{z_j}|^2 \geq 0.$$

□

We are ready to state the first main theorem.

Theorem 13. *Assume that $f(\mathbf{z})$ is a convenient non-degenerate holomorphic function. Consider a mixed homogeneous covering $\varphi = \varphi_{a,b} : \mathbb{C}^n \rightarrow \mathbb{C}^n$, $\varphi(\mathbf{w}, \bar{\mathbf{w}}) = (w_1^a \bar{w}_1^b, \dots, w_n^a \bar{w}_n^b)$ and let $g(\mathbf{w}, \bar{\mathbf{w}}) = f(\varphi(\mathbf{w}, \bar{\mathbf{w}}))$. Assume that $a > b > 0$ (respectively $0 < a < b$) and consider the link $K_r := g^{-1}(0) \cap \mathbb{S}_r$. Then there exists a positive number r_0 so that g is strictly holomorphic-like (resp. anti-holomorphic-like) on B_{r_0} and $K_r \subset \mathbb{S}_r$ is a positive (resp. negative) contact submanifold for any r , $0 < r \leq r_0$.*

If $f(\mathbf{z})$ is weighted homogeneous, $g(\mathbf{w}, \bar{\mathbf{w}})$ is strictly holomorphic-like (resp. anti-holomorphic-like) on \mathbb{C}^n and $K_r \subset \mathbb{S}_r$ is a positive contact submanifold for any $r > 0$.

Proof. By the convenience assumption of $f(\mathbf{z})$, $g(\mathbf{w}, \bar{\mathbf{w}})$ is convenient. As $f(\mathbf{z})$ has an isolated singularity at the origin and the restriction $\varphi : \mathbb{C}^{*I} \rightarrow \mathbb{C}^{*I}$ is a covering mapping for any $I \subset \{1, \dots, n\}$, g^I has an isolated mixed singularity at the origin. Put $g(\mathbf{w}, \bar{\mathbf{w}}) = \Re g(\mathbf{w}, \bar{\mathbf{w}}) + i \Im g(\mathbf{w}, \bar{\mathbf{w}})$. As a submanifold of \mathbb{C}^n , K is a complete intersection variety defined by three real valued functions $\rho = \Re g = \Im g = 0$. As the proof is completely the same, we assume that $a > b > 0$. To prove $\alpha \wedge d\alpha^{n-2}$ is positive non-vanishing on K_r , we can equivalently show that $d\rho \wedge \alpha \wedge d\alpha^{n-2} \wedge d\Re g \wedge d\Im g$ is locally non-vanishing on an arbitrary chosen point $\mathbf{z} \in K_r$ and positive. This follows from the fact that by the complete intersection property $\rho, \Re g, \Im g$ can be a part of real coordinate system of \mathbb{C}^n near any point of K_r . Namely there

exist real analytic functions h_4, \dots, h_{2n} such that $(\rho, \Re g, \Im g, h_4, \dots, h_{2n})$ are local coordinates. By Corollary 11 and Corollary, we have

$$d\rho \wedge \alpha \wedge d\alpha^{n-2} \wedge d\Re g \wedge d\Im g(\mathbf{w}, \bar{\mathbf{w}}) = i^n 2^{n-2} (n-2)! C(\mathbf{w}, \bar{\mathbf{w}}) dw_1 \wedge d\bar{w}_1 \wedge \dots \wedge dw_n \wedge d\bar{w}_n$$

The proof of the theorem is reduced to the following Lemma. \square

Lemma 14. (1) *A smooth link $K = g^{-1}(0) \cap \mathbb{S}_r$ is a contact submanifold of \mathbb{S}_r if and only if $C(\mathbf{w}, \bar{\mathbf{w}}) > 0$ on K .*

(2) *Assume that $g = \varphi^* f$ be as in Theorem 13 with $a > b > 0$. Then there exists a sufficiently small neighborhood U of the origin so that $C(\mathbf{w}, \bar{\mathbf{w}}) > 0$ for any $\mathbf{w} \in g^{-1}(0) \cap U \setminus \{\mathbf{0}\}$.*

If further $f(\mathbf{z})$ is weighted homogeneous, U can be the whole space \mathbb{C}^n .

Proof. Recall that

$$C_{j,k} = (a^2 - b^2) |w_j w_k|^2 \left(w_k^{a-1} \bar{w}_k^{b-1} \varphi^* f_{z_k} - w_j^{a-1} \bar{w}_j^{b-1} \varphi^* f_{z_j} \right)^2$$

Suppose that $C(\mathbf{w}, \bar{\mathbf{w}}) = 0$ for any small neighborhood. Applying the Curve Selection Lemma ([5]), we get a real analytic curve $\mathbf{w}(t)$, $0 \leq t \leq 1$ such that for any $1 \leq j, k \leq n$,

$$(11) \quad \begin{cases} w_j w_k \left(w_k^{a-1} \bar{w}_k^{b-1} \varphi^* f_{z_k} - w_j^{a-1} \bar{w}_j^{b-1} \varphi^* f_{z_j} \right) |_{\mathbf{w}=\mathbf{w}(t)} = 0, \\ g(\mathbf{w}(t), \bar{\mathbf{w}}(t)) \equiv 0, \quad \mathbf{w}(t) \in \mathbb{C}^n \setminus \{\mathbf{0}\}, \quad \mathbf{t} \neq \mathbf{0}. \end{cases}$$

Let $I = \{j \mid w_j(t) \neq 0\}$. Note that $|I| \geq 2$ as each coordinate axis is not included in $g^{-1}(0)$ by the convenience assumption. By the non-degeneracy assumption on f , there exists $j \in I$ such that $f_{z_j}(\varphi(\mathbf{w}(t), \bar{\mathbf{w}}(t))) \neq 0$. This implies by (11),

$$w_k(t) f_{z_k}(\varphi(\mathbf{w}(t), \bar{\mathbf{w}}(t))) \neq 0$$

for any $k \in I$. Take $k \in I$ and put

$$\gamma(t) = w_k^{a-1} \bar{w}_k^{b-1} \varphi^* f_{z_k} |_{\mathbf{w}=\mathbf{w}(t)}.$$

Then $\gamma(t) \neq 0$ and $\gamma(t)$ does not depend on the choice of $k \in I$. Put $v_j(t) := \frac{dw_j(t)}{dt}$ and take the differential of (11). As $v_j = 0$ for $j \notin I$, we get

$$\begin{aligned} 0 &= \frac{dg(\mathbf{w}(t), \bar{\mathbf{w}}(t))}{dt} \\ &= \sum_{j=1}^n f_{z_j}(\varphi(\mathbf{w}(t), \bar{\mathbf{w}}(t))) (a w_j(t)^{a-1} \bar{w}_j(t)^b v_j(t) + b w_j(t)^a \bar{w}_j(t)^{b-1} \bar{v}_j(t)) \\ &= \gamma(t) \sum_{j=1}^n (a \bar{w}_j(t) v_j(t) + b w_j(t) \bar{v}_j(t)) \\ &= \gamma(t) \frac{(a+b)}{2} \frac{d\|\mathbf{w}(t)\|^2}{dt}. \end{aligned}$$

Thus $\frac{d\|\mathbf{w}(t)\|^2}{dt} \equiv 0$. The last equality is derived from

$$\frac{d(\|\mathbf{w}(t)\|^2)}{dt} = \sum_{j=1}^n (w_j(t)\bar{v}_j(t) + \bar{w}_j(t)v_j(t)) = 2\Re \sum_{j=1}^n w_j(t)\bar{v}_j(t).$$

This implies that $\|\mathbf{w}(t)\|$ is constant which is a contradiction to the assumption $\|\mathbf{w}(t)\| \rightarrow 0$ ($t \rightarrow 0$).

We prove the second assertion. Assume that $f(\mathbf{z})$ is a weighted homogeneous polynomial of degree d with a weight vector $P = (p_1, \dots, p_n)$ and $a > b > 1$. Then $g = \varphi^* f$ is a strongly polar weighted homogeneous polynomial with $\text{rdeg}_P g = (a+b)d$ and $\text{pdeg}_P g = (a-b)d$. Put $I = \{j \mid w_j \neq 0\}$. Assume that $C_{j,k}(\mathbf{w}, \bar{\mathbf{w}}) = 0$ for any j, k for some $\mathbf{w} \in g^{-1}(0) \setminus \{0\}$. Put

$$\gamma = w_k^{a-1} \bar{w}_k^{b-1} \varphi^* f_{z_k}$$

for any fixed $k \in I$. Note that γ is independent of $k \in I$. As f^I (and also g^I) has an isolated singularity at the origin, $\gamma \neq 0$. Then this implies that

$$g_{w_j} = a w_j^{a-1} \bar{w}_j^b \varphi^* f_{z_j} = a \gamma \bar{w}_j, \quad j \in I$$

and by Euler equality (5), we get a contradiction

$$\begin{aligned} 0 &= a \, d \, g(\mathbf{w}, \bar{\mathbf{w}}) = \sum_{j=1}^n p_j w_j g_{w_j} \\ &= \sum_{j \in I} p_j w_j g_{w_j} \\ &= \gamma a \sum_{j \in I} p_j |w_j|^2 \neq 0. \end{aligned}$$

□

Corollary 15. *Assume that $f(\mathbf{z})$ is a holomorphic function with isolated singularity at the origin. Consider the link $K_r := g^{-1}(0) \cap \mathbb{S}_r$. Then there exists a positive number r_0 so that $K_r \subset \mathbb{S}_r$ is a positive contact submanifold for any $0 < r \leq r_0$.*

If $f(\mathbf{z})$ is weighted homogeneous, $K_r \subset \mathbb{S}_r$ is a positive contact submanifold for any $r > 0$.

Proof. The proof is parallel to that of Lemma 14. Recall that

$$C_{j,k} = |a \bar{z}_j f_{z_k} - a \bar{z}_k f_{z_j}|^2.$$

We do the same argument. If $\{\mathbf{z} \in \mathbb{C}^n \mid C(\mathbf{z}, \bar{\mathbf{z}}) = 0\} \cap f^{-1}(0)$ is not isolated at the origin, we take an analytic curve $\mathbf{z}(t) \in \{\mathbf{z} \mid C(\mathbf{z}, \bar{\mathbf{z}}) = 0\} \cap f^{-1}(0)$ as above. Putting $I = \{j \mid z_j(t) \neq 0\}$ as above, we get $|I| \geq 2$. As $f|_{\mathbb{C}^I}$ has an isolated singularity, we can assume that $f_{z_k}(\mathbf{z}(t)) \neq 0$ for some $k \in I$. Take $j \in I$ with $j \neq k$. Then $C_{j,k} = 0$ implies that $\bar{z}_j(t) f_{z_k}(\mathbf{z}(t)) \neq 0$ and thus $k \in I$ and $f_{z_j}(\mathbf{z}(t)) \neq 0$. Put $c(\mathbf{z}) = f_{z_k}(\mathbf{z})/\bar{z}_k$ for a fixed $k \in I$. This is a

non-zero and independent of $k \in I$. Note that $v_k(t) = 0$ for $k \notin I$. Therefore we get

$$\begin{aligned} 0 &= \frac{df(\mathbf{z}(t))}{dt} = \sum_{j=1}^n f_{z_j}(\mathbf{z}(t))v_j(t) \\ &= \sum_{j \in I} f_{z_j}(\mathbf{z}(t))v_j(t) = c(\mathbf{z}(t)) \sum_{j \in I} \bar{z}_j(t)v_j(t) \end{aligned}$$

and we get the same contradiction $\frac{d\|\mathbf{z}(t)\|^2}{dt} \equiv 0$.

Finally assume further $f(\mathbf{z})$ is weighted homogeneous of degree d with weight vector $P = (p_1, \dots, p_n)$. Put $I = \{j \mid z_j \neq 0\}$ as above and put $c(\mathbf{z}) = f_{z_k}(\mathbf{z})/\bar{z}_k$ for $k \in I$. Then $f_{z_j}(\mathbf{z}) = c(\mathbf{z})\bar{z}_j$ and we get the same contradiction:

$$0 = f(\mathbf{z}) = \sum_{j=1}^n p_j z_j f_{z_j}(\mathbf{z}) = \sum_{j=1}^n c(\mathbf{z}) p_j |z_j|^2 \neq 0.$$

Remark 16. *Corollary 15 gives a simple proof of holomorphic link to be a contact submanifold without using the strict pseudo-convex property.*

□

4. OPEN BOOK STRUCTURE.

4.1. Open book. *An open book with binding N on an oriented manifold M of dimension $2n - 1$ is a couple (N, θ) where N is a codimension two submanifold with a trivial normal bundle and $\theta : M \setminus N \rightarrow S^1$ is a local trivial smooth fibration where θ coincide with the angular coordinate of the trivial tubular neighborhood $N \times D_\delta \subset M$ ([4, 2]). The orientation of M gives a canonical orientation to the fiber $F_\eta := \theta^{-1}(\eta)$ for each $\eta \in S^1$. Restricting the fibration on $M \setminus N \times D_\delta$, the fiber $F'_\eta := F_\eta \cap (M \setminus N \times D_\delta)$ is a manifold with boundary N . Thus N has also a canonical orientation.*

4.2. Contact structure carried by an open book. Assume that we have a contact form ξ defined by a global 1-form α as before. We say that a contact structure ξ is *carried by an open book* (N, θ) if the following are satisfied ([4]).

- (1) The restriction of α to N is a contact form on N .
- (2) The two-form $d\alpha$ defines a symplectic form of each fiber $F_\eta = \theta^{-1}(\eta)$.
- (3) The orientation of N induced by α is the same as that of the boundary of F_η .

Recall that the condition (2) is equivalent to $d\theta(R) > 0$ where R is the Reeb vector field ([4, 3]). For further detail about a contact structure carried with an open book and symplectic structures, see H. Geiges [3], Giroux [4], R. Berndt [1] and Caubel-Nemethi-Popescu-Pampu [2].

4.3. Milnor open book for mixed functions. Let $g(\mathbf{z}, \bar{\mathbf{z}})$ be a convenient strongly non-degenerate mixed function. Let $V = g^{-1}(0)$ and we assume that V has an isolated mixed singularity at the origin. By Theorem 33 ([9]), we have

Theorem 17. *For a sufficiently small r , the mapping*

$$(12) \quad g/|g| : \mathbb{S}_r \setminus K_r \rightarrow S^1$$

is a locally trivial fibration.

By the transversality, we have a trivial tubular neighborhood $K_r \times D_\delta$ such that the following diagram commutes.

$$\begin{array}{ccccc} K_r \times D_\delta & \subset & K_r \times D_\delta^* & \subset & \mathbb{S}_r \setminus K_r \\ & & \downarrow p & \searrow \text{normal} & \downarrow g/|g| \\ & & D_\delta^* & & S^1 \end{array}$$

where $D_\delta^* = \{\eta \in \mathbb{C} \mid 0 \neq \eta, |\eta| \leq \delta\}$, p is the second projection and *normal* is the normalization map $\eta \mapsto \eta/|\eta|$. The argument θ is characterized by the equality:

$$\log g(\mathbf{z}, \bar{\mathbf{z}}) = \log |g(\mathbf{z}, \bar{\mathbf{z}})| + i\theta.$$

From this and the obvious equality $|g(\mathbf{z}, \bar{\mathbf{z}})|^2 = g(\mathbf{z}, \bar{\mathbf{z}})\bar{g}(\mathbf{z}, \bar{\mathbf{z}})$, we have

Proposition 18.

$$(13) \quad \nabla\theta = i \left(\frac{\overline{gz_1}}{\bar{g}} - \frac{gz_1}{g}, \dots, \frac{\overline{gz_n}}{\bar{g}} - \frac{gz_n}{g} \right),$$

$$(14) \quad d\theta = -i \left(\frac{\partial g + \bar{\partial}g}{g} - \frac{\partial \bar{g} + \bar{\partial}\bar{g}}{\bar{g}} \right),$$

where $\partial, \bar{\partial}$ are defined for a mixed function h by

$$dh = \partial h + \bar{\partial}h, \quad \partial h = \sum_{j=1}^n h_{z_j} dz_j, \quad \bar{\partial}h = \sum_{j=1}^n h_{\bar{z}_j} d\bar{z}_j.$$

4.4. Contact structure carried by a Milnor open book. We consider the existence problem of the contact structure carried by a Milnor open book for polar weighted homogeneous mixed functions. We consider a homogeneous mixed covering lifting $g(\mathbf{w}, \bar{\mathbf{w}}) = f(w_1^a \bar{w}_1^b, \dots, w_n^a \bar{w}_n^b)$ with $a > b \geq 0$ where $f(\mathbf{z})$ is a convenient non-degenerate holomorphic function defined in a neighborhood of the origin.

4.4.1. Strongly polar homogeneous case. First we consider the easy case that $f(\mathbf{z})$ is a homogeneous polynomial of degree d . Then $g(\mathbf{w}, \bar{\mathbf{w}})$ is a strongly polar homogeneous polynomial with $\text{rdeg } g = d(a+b)$, $\text{pdeg } g = d(a-b)$. In this case, we assert:

Theorem 19. *Assume that $f(\mathbf{z})$ is a homogeneous polynomial with $\text{pdeg } f = d > 0$. The canonical contact form α and ω is adapted with the Milnor open book $\theta : S_r^{2n-1} \setminus K_r \rightarrow S^1$ for any $r > 0$.*

Proof. Recall that Reeb vector field R and $d\theta$ are given on \mathbb{S}_r by

$$R(\mathbf{z}) = \frac{i}{2r^2} \sum_{j=1}^n \left(z_j \frac{\partial}{\partial z_j} - \bar{z}_j \frac{\partial}{\partial \bar{z}_j} \right)$$

$$d\theta = -i \left\{ \frac{\partial g + \bar{\partial} g}{g} - \frac{\partial \bar{g} + \bar{\partial} \bar{g}}{\bar{g}} \right\}$$

In fact, we use the polar Euler equality:

$$\sum_{j=1}^n (z_j g_{z_j} - \bar{z}_j g_{\bar{z}_j}) = d(a-b)g$$

and its conjugate:

$$\sum_{j=1}^n (\bar{z}_j \bar{g}_{\bar{z}_j} - z_j \bar{g}_{z_j}) = d(a-b)\bar{g}.$$

Using these Euler equalities, we get:

$$\begin{aligned} d\theta(R) &= \frac{\partial g(R) + \bar{\partial} g(R)}{g} - \frac{\partial \bar{g}(R) + \bar{\partial} \bar{g}(R)}{\bar{g}} \\ &= \sum_{j=1}^n \frac{z_j g_{z_j} - \bar{z}_j g_{\bar{z}_j}}{g} - \sum_{j=1}^n \frac{z_j \bar{g}_{z_j} - \bar{z}_j \bar{g}_{\bar{z}_j}}{\bar{g}} \\ &= 2d(a-b) > 0. \end{aligned}$$

This shows that for any radius $r > 0$, the canonical contact structure on $K_r \subset \mathbb{S}_r$ is adapted with the Milnor fibration. \square

5. GENERAL CASE

We are interested in the existence of open book structure adapted to the contact structure which is the restriction of α to the link $K_r \subset \mathbb{S}_r$ where $K_r = g^{-1}(0) \cap \mathbb{S}_r$. We have shown that there exists a canonical Milnor fibration on $g/|g| : \mathbb{S}_r \setminus \{K_r\} \rightarrow S^1$ by [9]. However this fibration is not adapted with the symplectic structure given by $d\alpha$. Therefore we will change the contact form α without changing the contact structure ξ so that the new contact form will be carried by the Milnor open book. We follow the proof of Theorem 3.9 in [2] for the holomorphic functions in Caubel-Némethi-Popescu-Pampu.

We modify the contact form α by

$$\alpha_c = e^{-c|g|^2} \alpha$$

with a sufficiently large positive real number $c > 0$. This does not change the contact structure $\xi = \text{Ker } \alpha$ but the two form $\omega_c = d\alpha_c$ is changed as

$$\omega_c = d(e^{-c|g|^2}) \wedge \alpha + e^{-c|g|^2} d\alpha$$

and the corresponding symplectic structure changes. Consider the new Reeb vector field R_c . Put $H = e^{-c|g|^2}$. Put also $R_c = k(R + S_c)$ with S_c is tangent to ξ . Then we get $kH = 1$. As

$$d\alpha_c = dH \wedge \alpha + H d\alpha,$$

the condition for R_c to be the Reeb vector field $\iota_{R_c} d\alpha_c|_\xi = 0$ gives the condition:

$$(15) \quad \iota_{S_c} \omega|_\xi = \frac{dH}{H}|_\xi = -c d|g|^2.$$

Put $\pi : T_{\mathbf{w}}\mathbb{C}^n \rightarrow \xi(\mathbf{w})$ be the hermitian orthogonal projection. Namely, $\pi(\mathbf{v}) = \mathbf{v} - (\mathbf{v}, \tilde{R})\tilde{R}$ and $\tilde{R} = R/\|R\|$. Then (15) implies by (8) that

$$\begin{aligned} \iota_{S_c} \omega|_\xi &= -c d|g|^2|_\xi = \Re(-c \nabla |g|^2, \cdot)|_\xi \\ &= \frac{1}{4} \omega(ic \nabla |g|^2, \cdot)|_\xi = \omega(\pi(ic \nabla |g|^2)/4, \cdot)|_\xi \\ &= \iota_{\pi(ic \nabla |g|^2)/4} \omega|_\xi. \end{aligned}$$

As ω is non-degenerate on ξ , we get

$$(16) \quad S_c = \pi(ic \nabla |g|^2/4)$$

Thus we get

$$(17) \quad |g|^2 d\theta(R_c) = k|g|^2 d\theta(R) + k\Re(|g|^2 \nabla \theta, S_c)$$

$$(18) \quad = k|g|^2 d\theta(R) + k\Re(\pi(|g|^2 \nabla \theta), \pi(ic \nabla |g|^2/4)).$$

For simplicity, we introduce two vectors

$$(19) \quad \nabla \partial g = (g_{w_1}, \dots, g_{w_n})$$

$$(20) \quad \nabla \bar{\partial} g = (g_{\bar{w}_1}, \dots, g_{\bar{w}_n}).$$

Remark 20. In our previous paper [8], we used the notation dg and $\bar{d}g$ instead of $\nabla \partial g$ and $\nabla \bar{\partial} g$. We changed notations as the previous notations are confusing with 1-forms ∂g , $\bar{\partial} g$. We use dg not for ∂g but $dg = (\partial + \bar{\partial})g$.

Recall that

$$(21) \quad \nabla |g|^2 = 2g \overline{\nabla \partial g} + 2\bar{g} \nabla \bar{\partial} g$$

$$(22) \quad |g|^2 \nabla \theta = ig \overline{\nabla \partial g} - i\bar{g} \nabla \bar{\partial} g.$$

Thus $2|g|^2 \nabla \theta$ and $i \nabla |g|^2$ are different in the case of mixed functions. This makes a difficulty. (In the holomorphic function case, they are the same up to a scalar multiplication, as $\nabla \bar{\partial} g$ vanishes.)

Put $\pi' : \mathbb{C}^n \rightarrow \mathbb{C} \cdot R$ be the orthogonal projection to the complex line $\mathbb{C} \cdot R = \mathbb{C} \cdot \mathbf{w}$ generated by R or \mathbf{w} . Namely $\pi'(\mathbf{v}) = (\mathbf{v}, \mathbf{w})\mathbf{w}/\|\mathbf{w}\|^2 =$

$(\mathbf{v}, R)R/\|R\|^2$. Then $\pi(\mathbf{v}) = \mathbf{v} - \pi'(\mathbf{v})$. Consider the expression:

$$(23) \quad \overline{g \nabla \partial g} = \mathbf{v}_{11} + \mathbf{v}_{12}, \quad \begin{cases} \mathbf{v}_{11} = \pi(g \overline{\nabla \partial g}) \\ \mathbf{v}_{12} = \pi'(g \overline{\nabla \partial g}) \end{cases}$$

$$(24) \quad \bar{g} \nabla \bar{\partial} g = \mathbf{v}_{21} + \mathbf{v}_{22}, \quad \begin{cases} \mathbf{v}_{21} = \pi(\bar{g} \nabla \bar{\partial} g) \\ \mathbf{v}_{12} = \pi'(\bar{g} \nabla \bar{\partial} g). \end{cases}$$

Using this expression, we get

$$(25) \quad \pi(ic \nabla |g|^2/4) = \frac{ic}{2}(\mathbf{v}_{11} + \mathbf{v}_{21})$$

$$(26) \quad \pi(|g|^2 \nabla \theta) = i(\mathbf{v}_{11} - \mathbf{v}_{21})$$

Thus we get

$$(27) \quad \begin{aligned} |g|^2 d\theta(R_c) &= k|g|^2 d\theta(R) + k\Re(\pi(|g|^2 \nabla \theta), \pi(ic \nabla |g|^2/2)) \\ &= k|g|^2 d\theta(R) + \frac{ck}{2}(\|v_{11}\|^2 - \|v_{21}\|^2). \end{aligned}$$

Here we have used the equality: $\Re(\mathbf{v}_{21}, \mathbf{v}_{11}) = \Re(\mathbf{v}_{11}, \mathbf{v}_{21})$. The key assertion is the following.

Lemma 21. *We have the inequality: $\|v_{11}\|^2 - \|v_{21}\|^2 \geq 0$ and the equality takes place if and only if $\overline{\nabla \partial g}(\mathbf{w}, \bar{\mathbf{w}}) = \lambda_1 \mathbf{w}$ and $\nabla \bar{\partial} g(\mathbf{w}, \bar{\mathbf{w}}) = \lambda_2 \mathbf{w}$ for some $\lambda_1, \lambda_2 \in \mathbb{C}$. In this case, we have also $\nabla \theta = \lambda R$ for some $\lambda \in \mathbb{C}$.*

Proof. Let $\mathbf{v}_1 = \overline{g \nabla \partial g}$ and $\mathbf{v}_2 = \bar{g} \nabla \bar{\partial} g$. As $\{\mathbf{v}_{11}, \mathbf{v}_{12}\}$ and $\{\mathbf{v}_{12}, \mathbf{v}_{22}\}$ are hermitian orthogonal, we have

$$\|\mathbf{v}_{11}\|^2 = \|\mathbf{v}_1\|^2 - \|\mathbf{v}_{12}\|^2, \quad \|\mathbf{v}_{21}\|^2 = \|\mathbf{v}_2\|^2 - \|\mathbf{v}_{22}\|^2.$$

We go now further precise expression. Put

$$\mathbf{v}_1 = (v_1^1, \dots, v_1^n), \quad \mathbf{v}_2 = (v_2^1, \dots, v_2^n).$$

Then we have

$$\begin{aligned} v_1^j &= g(\mathbf{w}, \bar{\mathbf{w}}) \overline{f_{z_j}(\varphi(\mathbf{w}, \bar{\mathbf{w}}))} a \bar{w}_j^{a-1} w_j^b \\ v_2^j &= \bar{g}(\mathbf{w}, \bar{\mathbf{w}}) f_{z_j}(\varphi(\mathbf{w}, \bar{\mathbf{w}})) b w_j^a \bar{w}_j^{b-1}. \end{aligned}$$

Recall that $R = i\mathbf{w}/2\rho(\mathbf{w})$. Thus

$$\begin{aligned}
\mathbf{v}_{12} &= \left(\sum_{j=1}^n g(\mathbf{w}, \bar{\mathbf{w}}) \overline{f_{z_j}(\varphi(\mathbf{w}, \bar{\mathbf{w}}))} a \bar{w}_j^a w_j^b \right) \mathbf{w} / \|\mathbf{w}\|^2 \\
&= a g(\mathbf{w}, \bar{\mathbf{w}}) \left(\sum_{j=1}^n \overline{f_{z_j}(\varphi(\mathbf{w}, \bar{\mathbf{w}}))} \bar{w}_j^a w_j^b \right) \mathbf{w} / \|\mathbf{w}\|^2 \\
\mathbf{v}_{22} &= \left(\sum_{j=1}^n \bar{g}(\mathbf{w}, \bar{\mathbf{w}}) f_{z_j}(\varphi(\mathbf{w}, \bar{\mathbf{w}})) b w_j^a \bar{w}_j^b \right) \mathbf{w} / \|\mathbf{w}\|^2 \\
&= \bar{b}(\mathbf{w}, \bar{\mathbf{w}}) g(\mathbf{w}, \bar{\mathbf{w}}) \left(\sum_{j=1}^n f_{z_j}(\varphi(\mathbf{w}, \bar{\mathbf{w}})) w_j^a \bar{w}_j^b \right) \mathbf{w} / \|\mathbf{w}\|^2.
\end{aligned}$$

Thus we get

$$\begin{aligned}
0 \leq \|\mathbf{v}_{11}\|^2 &= \|\mathbf{v}_1\|^2 - \|\mathbf{v}_{12}\|^2 \\
&= a^2 |g|^2 \sum_{j=1}^n |f_{z_j}|^2 |w_j|^{2(a+b-1)} - a^2 |g|^2 \left| \sum_{j=1}^n \overline{f_{z_j}} \bar{w}_j^a w_j^b \right|^2 \\
&= a^2 |g|^2 (\gamma - \beta) \\
0 \leq \|\mathbf{v}_{21}\|^2 &= \|\mathbf{v}_2\|^2 - \|\mathbf{v}_{22}\|^2 \\
&= b^2 |g|^2 \sum_{j=1}^n |f_{z_j}|^2 |w_j|^{2(a+b-1)} - b^2 |g|^2 \left| \sum_{j=1}^n f_{z_j} w_j^a \bar{w}_j^b \right|^2 \\
&= b^2 |g|^2 (\gamma - \beta) \\
&\text{where } \begin{cases} \gamma &= \sum_{j=1}^n |f_{z_j}|^2 |w_j|^{2(a+b-1)} \\ \beta &= \left| \sum_{j=1}^n \overline{f_{z_j}} \bar{w}_j^a w_j^b \right|^2. \end{cases}
\end{aligned}$$

Thus $\gamma \geq \beta$ and we have

$$\|\mathbf{v}_{11}\|^2 - \|\mathbf{v}_{21}\|^2 = (a^2 - b^2) |g|^2 (\gamma - \beta) \geq 0$$

and the equality holds if and only if $\gamma = \beta$. This is equivalent to $\|\mathbf{v}_{11}\| = \|\mathbf{v}_{21}\| = 0$ and this implies $\overline{\nabla_{\partial} g}(\mathbf{w}, \bar{\mathbf{w}}) = \lambda_1 \mathbf{w}$ and $\nabla_{\bar{\partial}} g(\mathbf{w}, \bar{\mathbf{w}}) = \lambda_2 \mathbf{w}$ for some $\lambda_1, \lambda_2 \in \mathbb{C}$. The last assertion follows from $|g|^2 \nabla \theta = i(\mathbf{v}_1 - \mathbf{v}_2)$. \square

5.1. Main theorem. Now we are ready to state our main theorem. Let $\varphi(\mathbf{w}, \bar{\mathbf{w}}) = (w_1^a \bar{w}_1^b, \dots, w_n^a \bar{w}_n^b)$ with $a > b > 0$ as before.

Main Theorem 22. *Assume that $f(\mathbf{z})$ is a convenient non-degenerate holomorphic function so that $g(\mathbf{w}, \bar{\mathbf{w}}) = \varphi^* f(\mathbf{w}, \bar{\mathbf{w}})$ is a convenient non-degenerate mixed function of strongly polar weighted homogeneous face type.*

Then there exists a positive number r_0 such that the Milnor open book $f/|f|: \mathbb{S}_r \setminus K_r \rightarrow S^1$ carries a contact structure for any $r > 0$ with $r \leq r_0$.

If further $f(\mathbf{z})$ is a weighted homogeneous polynomial, we can take $r_0 = \infty$ and any $r > 0$.

Proof. For the proof, we do the same discussion as that of Caubel-Némethi-Popescu-Pampu [2]. Let

$$Z_\delta := \{\mathbf{w} \in S_r^{2n-1} \setminus V_\delta \mid d\theta(R) \leq 0\}, \quad V_\delta = S_r^{2n-1} \cap g^{-1}(D_\delta)$$

where δ is sufficiently small so that $f^{-1}(0)$ and \mathbb{S}_r are transverse and V_δ is a trivial tubular neighborhood. Let α_c and R_c be as before. As we have shown that

$$|g|^2 d\theta(R_c) = k|g|^2 d\theta(R) + \frac{ck}{2}(\|v_{11}\|^2 - \|v_{21}\|^2)$$

with $k = 1/e^{-c|g|^2}$ and the second term is non-negative and the equality holds (i.e. $\|v_{11}\|^2 - \|v_{21}\|^2 = 0$) if and only if $\nabla\theta = \lambda R$. In this case, $d\theta(R) = \Re\lambda\|R\|^2$ and $d\theta(R)$ is positive if $\Re\lambda > 0$. Thus taking sufficiently large $c > 0$, we only need to show that $\nabla\theta$ and R are linearly independent on Z_δ . Thus the following lemma completes the proof. Compare with Proposition 3.8 ([2]). \square

Lemma 23. *Assume that $\nabla\theta(\mathbf{w}) = \lambda R(\mathbf{w})$ on $\mathbf{w} \in Z_\delta$ for some $\lambda \in \mathbb{C}$.*

- (1) *Assume that $f(\mathbf{z})$ is convenient non-degenerate weighted homogeneous polynomial. Then λ is a positive real number.*
- (2) *If $f(\mathbf{z})$ is not weighted homogeneous but a convenient non-degenerate mixed function of strongly polar weighted homogeneous face type, there exists a positive number r_0 such that $\Re\lambda$ is positive for any $\mathbf{w} \in \mathbb{S}_r \setminus g^{-1}(0)$ and $r \leq r_0$.*

Proof. (1) Assume that $f(\mathbf{z})$ is a weighted homogeneous of degree d with weight vector $P = (p_1, \dots, p_n)$. Let $m_r = (a+b)d$ and $m_p = (a-b)d$, the radial and polar degree of $g(\mathbf{w}, \bar{\mathbf{w}})$. The assumption says that

$$(28) \quad \lambda w_j = \frac{1}{\bar{g}(\mathbf{w}, \bar{\mathbf{w}})} \overline{g_{w_j}(\mathbf{w}, \bar{\mathbf{w}})} - \frac{1}{g(\mathbf{w}, \bar{\mathbf{w}})} g_{\bar{w}_j}(\mathbf{w}, \bar{\mathbf{w}})$$

for $j = 1, \dots, n$. Taking the summation of (28) $\times p_j \bar{w}_j$ for $j = 1, \dots, n$, we get:

$$\begin{aligned} \lambda \sum_{j=1}^n p_j |w_j|^2 &= \frac{1}{\bar{g}(\mathbf{w}, \bar{\mathbf{w}})} \sum_{j=1}^n p_j \bar{w}_j \overline{g_{w_j}(\mathbf{w}, \bar{\mathbf{w}})} - \frac{1}{g(\mathbf{w}, \bar{\mathbf{w}})} \sum_{j=1}^n p_j \bar{w}_j g_{\bar{w}_j}(\mathbf{w}, \bar{\mathbf{w}}) \\ &= (m_r + m_p) - (m_r - m_p) = 2m_p > 0 \end{aligned}$$

by the strong Euler equalities (5). This implies λ is a positive number.

(2) **General case.** Assume that $g(\mathbf{w}, \bar{\mathbf{w}})$ is a convenient non-degenerate mixed function of strongly polar weighted homogeneous face type. Assume that the assertion (2) does not hold. Using Curve Selection Lemma ([6, 5]),

we can find a real analytic curve $\mathbf{w}(t) \in \mathbb{C}^n \setminus f^{-1}(0)$ for $0 < t \leq \varepsilon$ and Laurent series $\lambda(t)$ such that

$$(29) \quad \nabla \theta(\mathbf{w}(t)) = \lambda(t) R(\mathbf{w}(t))$$

such that $\nabla \theta(R(\lambda(t))) \leq 0$. We show this give a contradiction by showing $\lim_{t \rightarrow 0} \arg \lambda(t) = 0$.

Let $I = \{j \mid w_j(t) \neq 0\}$. Then $|I| \geq 2$ and we restrict our the discussion to the coordinate subspace \mathbb{C}^I and $g^I = g|_{\mathbb{C}^I}$. For the notation's simplicity, we assume that $I = \{1, \dots, n\}$ hereafter. Consider the Taylor (Laurent) expansions:

$$\begin{aligned} w_j(t) &= a_j t^{p_j} + (\text{higher terms}), \quad j = 1, \dots, n, \\ \lambda(t) &= \lambda_0 t^\ell + (\text{higher terms}), \\ g(\mathbf{w}(t), \bar{\mathbf{w}}(t)) &= g_0 t^d + (\text{higher terms}) \end{aligned}$$

where $a_j, \lambda_0, g_0 \neq 0$ and $p_j \in \mathbb{N}$, $\ell \in \mathbb{Z}$. Then the equality (29) says

$$(30) \quad \lambda(t) w_j(t) = \frac{\overline{g_{w_j}(\mathbf{w}(t), \bar{\mathbf{w}}(t))}}{\bar{g}(\mathbf{w}(t), \bar{\mathbf{w}}(t))} - \frac{g_{\bar{w}_j}(\mathbf{w}(t), \bar{\mathbf{w}}(t))}{g(\mathbf{w}(t), \bar{\mathbf{w}}(t))},$$

for $j = 1, \dots, n$. Consider the weight vector $P = (p_1, \dots, p_n)$ and the face function f_P . Then $g_P = \varphi^* f_P$. By (30) we get the equalities:

$$(31) \quad \lambda_0 a_j t^{p_j + \ell} + \dots = \left(\frac{\overline{(g_P)_{w_j}(\mathbf{a})}}{\bar{g}_0} - \frac{(g_P)_{\bar{w}_j}(\mathbf{a})}{g_0} \right) t^{d(P; f) - p_j - d} + \dots$$

where $j = 1, \dots, n$ and $\mathbf{a} = (a_1, \dots, a_n)$. The order of the left side for j is $p_j + \ell$. The order of the right side is at least $d(P; f) - p_j - d$. Thus we have

$$(32) \quad p_j + \ell \geq d(P; f) - p_j - d.$$

Put

$$C_j := \left(\frac{\overline{(g_P)_{w_j}(\mathbf{a})}}{\bar{g}_0} - \frac{(g_P)_{\bar{w}_j}(\mathbf{a})}{g_0} \right).$$

The equality in (32) holds if $C_j \neq 0$:

$$(33) \quad p_j + \ell = d(P, g) - p_j - d, \quad \text{if } C_j \neq 0.$$

We assert that

Assertion 24. *There exists some j such that $C_j \neq 0$.*

Proof. Assume that $C_1 = \dots = C_n = 0$. This implies that $\overline{\nabla_{\partial} g(\mathbf{a})} = u \nabla_{\bar{\partial}} g(\mathbf{a})$ with $u = \bar{b}/b$ and therefore we see that \mathbf{a} is a critical point of $g_P : \mathbb{C}^{*n} \rightarrow \mathbb{C}$ by Proposition 1 ([8]) which contradicts to the non-degeneracy assumption. \square

Let $p_{\min} = \min \{p_j \mid j = 1, \dots, n\}$ and $J = \{j \mid p_j = p_{\min}\}$ and let $p_{\max} = \max \{p_j \mid C_j \neq 0\}$ and $J' = \{j \mid p_j = p_{\max}, C_j \neq 0\}$. We assert that

Assertion 25. $p_{\min} = p_{\max}$.

Proof. Assume that $p_{\min} < p_{\max}$. Then we have a contradiction: For $k \in J$ and $j \in J'$,

$$p_k + \ell < p_j + \ell = d(P, g) - p_j - d < d(P, g) - p_k - d$$

which contradicts to (33). \square

Thus we have proved the equivalence $C_j = 0 \iff j \notin J$ and comparing the leading coefficients of (31),

$$(34) \quad \lambda_0 a_k = C_k, \quad p_{\min} + \ell = d(P, g) - p_{\min} - d, \quad \forall k \in J.$$

Then taking the summation $\sum_{j \in J} p_j \bar{a}_j \times (34)$, we get the equality

$$(35) \quad \sum_{k \in J} p_k \lambda_0 \bar{a}_k a_k = \sum_{k \in J} p_k \bar{a}_k C_k.$$

The left side is $\lambda_0 \sum_{k \in J} p_k |a_k|^2 \neq 0$. The right side is

$$\begin{aligned} \sum_{k \in J} p_k \bar{a}_k C_k &= \sum_{k=1}^n p_k \bar{a}_k C_k \\ &= \sum_{k=1}^n p_k \bar{a}_k \left(\overline{(g_P)_{w_j}}(\mathbf{a}, \bar{\mathbf{a}}) / \bar{g}_0 - (g_P)_{\bar{w}_j}(\mathbf{a}, \bar{\mathbf{a}}) / g_0 \right) \\ &= (\text{rdeg}(P, g_P) + \text{pdeg}(P, g_P)) \overline{g_P}(\mathbf{a}, \bar{\mathbf{a}}) / \bar{g}_0 \\ &\quad - (\text{rdeg}(P, g_P) - \text{pdeg}(P, g_P)) g_P(\mathbf{a}, \bar{\mathbf{a}}) / g_0. \end{aligned}$$

As the left side is non-zero, we have $g_P(\mathbf{a}, \bar{\mathbf{a}}) \neq 0$ and $d = d(P, g)$ and $g_0 = g_P(\mathbf{a}, \bar{\mathbf{a}})$. Thus we finally obtain the equality

$$\lambda_0 \sum_{k \in J} p_k |a_k|^2 = 2 \text{pdeg}(P, g_P)$$

which implies $\lambda_0 > 0$ and thus $\lim_{t \rightarrow 0} \arg \lambda(t) = 0$. As $d\theta(R(\mathbf{w}(t))) = \Re \nabla \theta(R(\mathbf{w}(t))) = \Re \lambda(t) \|R(\mathbf{w}(t))\|^2 > 0$ for a sufficiently small t , this is a contradiction. \square

Remark 26. 1. Lemma 23 hold for any convenient non-degenerate mixed function $g(\mathbf{w}, \bar{\mathbf{w}})$ of strongly polar weighted homogeneous face type, as the proof do not use the assumption $g = \varphi^* f$.

2. Let $g = \varphi^* f$ where $f(\mathbf{z})$ is a convenient non-degenerate holomorphic function and $\varphi_{a,b}$ is a homogeneous cyclic covering map. Then

Assertion 27. The link topology of $g^{-1}(0)$ is a combinatorial invariant and it is determined by $\Gamma(f)$.

Proof. Assume that $f'(\mathbf{z})$ is another convenient non-degenerate holomorphic function with $\Gamma(f') = \Gamma(f)$ and let $g' = \varphi^* f'$. Take a one-parameter family $f_t(\mathbf{z})$, $0 \leq t \leq 1$ so that $\Gamma(f_t) = \Gamma(f)$, $f_0 = f$, $f_1 = f'$ and $f_t(\mathbf{z})$ is non-degenerate for any t . Then we get a one-parameter family $g_t := \varphi^* f_t$ of mixed function of strongly polar weighted homogeneous face type. Then their links are certainly isotopic. \square

Observe that there exists in general mixed functions $h(\mathbf{w}, \bar{\mathbf{w}})$ which is convenient, non-degenerate and of strongly polar weighted homogeneous face type but it is not a homogenous lift of a holomorphic function. In such a case, the topology of the links of g and h may be different. See Example 5.4 in [7].

REFERENCES

- [1] R. Berndt. *An introduction to symplectic geometry*, volume 26 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001. Translated from the 1998 German original by Michael Klucznik.
- [2] C. Caubel, A. Némethi, and P. Popescu-Pampu. Milnor open books and Milnor fillable contact 3-manifolds. *Topology*, 45(3):673–689, 2006.
- [3] H. Geiges. *An introduction to contact topology*, volume 109 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2008.
- [4] E. Giroux. Contact structures and symplectic fibrations over the circle.
- [5] H. Hamm. Lokale topologische Eigenschaften komplexer Räume. *Math. Ann.*, 191:235–252, 1971.
- [6] J. Milnor. *Singular points of complex hypersurfaces*. Annals of Mathematics Studies, No. 61. Princeton University Press, Princeton, N.J., 1968.
- [7] M. Oka. Mixed functions of strongly polar weighted homogeneous face type, arxiv 1202.2166v1.
- [8] M. Oka. Topology of polar weighted homogeneous hypersurfaces. *Kodai Math. J.*, 31(2):163–182, 2008.
- [9] M. Oka. Non-degenerate mixed functions. *Kodai Math. J.*, 33(1):1–62, 2010.

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